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On optimum design in fluid mechanics

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In this paper, the change in energy dissipation due to a small hump on a body in a uniform steady flow is calculated. The result is used in conjunction with the variational methods of optimal control to obtain the optimality conditions for four minimum-drag problems of fluid mechanics. These conditions imply that the unit-area profile of smallest drag has a front end shaped like a wedge of angle 90°.

1. Introduction

What is the shape of the body (of, say, given volume) which has minimum drag when moved at constant speed in a viscous fluid? Although engineers have tried to answer similar questions for many years, not much is known theoretically about such bodies. At high Reynolds number they must be slender so that the boundary layer does not separate. In a previous (Pironneau 1973) we have shown that the variational methods used in optimal control can be of great help for such problems. However, owing to the current state of optimal control theory, only the case of low Reynolds number was considered, the flow being described by linear equations. We found that the unit-volume body with smallest drag has uniform skin friction and is shaped like a rugby ball, with conical front and rear ends of angle 120°. This result is a consequence of the fact that in Stokes flow the change in energy dissipation due to a hump of height $\alpha(\mathbf{s})$ on

$$S = \{\boldsymbol{\xi}(\mathbf{s}) | \mathbf{s} \in [0, 1]^2\}$$

is

$$\delta E = \nu \int_S \alpha(\mathbf{s}) \left\| \frac{\partial \mathbf{u}}{\partial n} \right\|^2 dS + o(\alpha(\,)), \tag{1.1}$$

where $\mathbf{u}(\mathbf{x})$ is the fluid velocity vector.

In this paper, we shall essentially generalize this result to laminar flow at higher Reynolds number. We shall show that, in some sense, the steady Navier–Stokes equations can be linearized about $\mathbf{u}(\mathbf{x})$ and that the same variational methods of optimal control can then be used to derive a formula similar to (1.1). We obtain

$$\delta E = \nu \int_S \alpha(\mathbf{s}) \left(\left\| \frac{\partial \mathbf{u}}{\partial n} \right\|^2 + 2 \frac{\partial \mathbf{u}}{\partial n} \cdot \frac{\partial \mathbf{w}}{\partial n} \right) dS, \tag{1.2}$$

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where $\mathbf{u}$, together with q, is a solution of†

$$\nu\nabla^2\mathbf{w} - \nabla\mathbf{u}.\mathbf{w} + \mathbf{u}.\nabla\mathbf{w} = -\mathbf{u}.\nabla\mathbf{u} + \nabla q, \quad \nabla.\mathbf{w} = 0; \quad \mathbf{w}|_S = 0, \quad \mathbf{w}|_\infty = 0. \quad (1.3)$$

In optimal control theory, $\mathbf{w}$ is known as the co-state vector of $\mathbf{u}$. The unit-volume body with smallest drag must then be such that

$$\left\|\frac{\partial\mathbf{u}}{\partial n}\right\|^2 + 2\frac{\partial\mathbf{w}}{\partial n}.\frac{\partial\mathbf{u}}{\partial n} \text{ is constant on } S. \quad (1.4)$$

It is unfortunate that (1.2) contains an element like $\mathbf{w}$ for which there is no simple mechanical interpretation. Equation (1.4) must simply be looked upon as the equation of the body with smallest drag; its complexity reflects the complexity of the problem. Moreover, as pointed out in the previous paper, in cases where the steady Navier–Stokes equation can be solved numerically, there is a natural way of solving (1.4) on a computer. Thus the derivation of equations like (1.4) constitutes the necessary mathematical analysis preceding numerical solution of the problem. We shall also derive similar equations (known in optimal control as 'necessary optimality conditions') for the following problems.

(i) Minimum-drag body of given surface area.
(ii) Minimum-drag shell of a given body.
(iii) Minimum-drag profile for a given lift.

All these cases require that $\partial\mathbf{w}/\partial n$ be known. However, since $\mathbf{w}$ is described by an equation very similar to the one that describes $\mathbf{u}$, we can gain some knowledge of $\partial\mathbf{w}/\partial n$ via boundary-layer theory. This is done in §4, where we show that a wedge of angle 90° satisfies (1.4); that is, the two-dimensional unit-area body with smallest drag has such a wedge-shaped front end.

This paper is organized such that the mathematical justification of the preceding results is at the end. Section 2 deals with the statement of the problem; then §3 states the main theorem concerning the change in energy dissipation due to a small hump, as well as its consequences concerning the optimality conditions of the previous problems. In §4 we discuss the meaning of the equations obtained, and show that they are compatible with those obtained in the previous paper for low Reynolds number flow. The main theorem is proved in §5.

2. Statement of the problem

Consider the optimal control problem

$$\min_{S\in\mathscr{S}}\left\{\frac{2}{\nu}\int_{\Omega_S}\sigma_{ij}\sigma_{ij}\,d\Omega \,\middle|\, \nu\nabla^2\mathbf{u} - \mathbf{u}.\nabla\mathbf{u} = \nabla p, \quad \nabla.\mathbf{u} = 0 \quad \text{in} \quad \Omega_S; \quad \mathbf{u}|_S = 0, \quad \mathbf{u}|_\Gamma = \mathbf{g}\right\}, \quad (2.1)$$

where $\sigma_{ij} = \frac{1}{2}\nu(\partial u_i/\partial x_j + \partial u_j/\partial x_i)$ $(i,j = 1,\dots,n;\ n = 2 \text{ or } 3)$; $\mathscr{S}$ is a given subset of the set of bounded smooth‡ surfaces in R^n; Ω_S is the bounded open set of R^n

† $-\nabla\mathbf{u}.\mathbf{w} + \mathbf{u}.\nabla\mathbf{w}_i = w_j\,\partial u_j/\partial x_i - u_j\,\partial w_i/\partial x_j$.

‡ S, $\Gamma \in \Lambda_{2,h}$, $g \in C_{2,h}(\Gamma)$, and S, Γ and $\mathbf{g}$ must be such that $(\mathbf{g}|\Gamma, 0|S)$ can be extended into a solenoidal field in Ω_S and that there exist cut-off functions $\xi(x,\delta)$ near (S,Γ). (See Ladyzhenskaya 1963.)

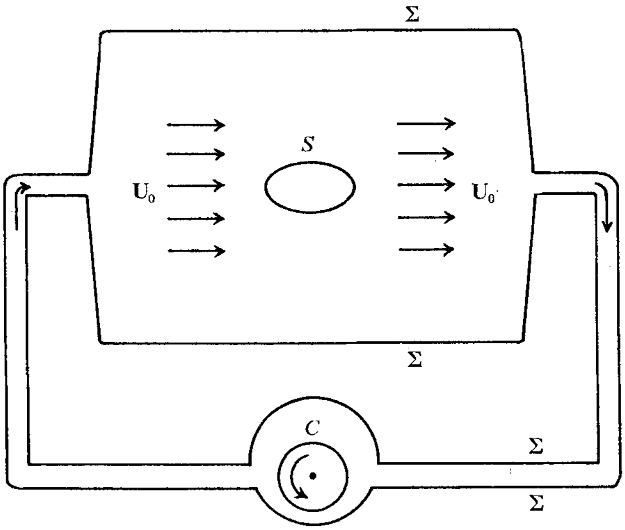

FIGURE 1. A possible design for the study of the drag on S. The fluid inside Σ is maintained in motion by the pump C (a rotating cylinder for example). S is very small compared with Σ so that the motion is almost uniform far from S. In this case, $\Gamma = \Sigma \cup C$, $\mathbf{z} = 0$ on Σ, $\mathbf{z} = \mathbf{w} \times \mathbf{x}$, at $\mathbf{x} \in C$, where ω is the angular velocity of C.

with boundary $\partial\Omega_S = S \cup \Gamma$; $\mathbf{u} = (u_1, \ldots, u_m)$ is the solution (in the sense of theorem 5.2.4 in Ladyzhenskaya 1963) of the steady Navier–Stokes equations in Ω_S; and $\mathbf{g}$ is a given smooth velocity profile on Γ with zero flux $\left(\int_\Gamma \mathbf{g} \, . \, d\Gamma = 0\right)$. It is shown in Ladyzhenskaya, under fairly general assumptions on S, Γ and $\mathbf{g}$, that the differential system above has at least one solution $(\mathbf{u}, p)$ with $\mathbf{u}$ almost everywhere twice differentiable in Ω_S. Furthermore, it was shown by Finn (1959; see also Heywood 1970) that if ν is not too small, or if the domain Ω_S is such that

$$\nu > n^{\frac{1}{2}} D_\Omega \|\nabla \mathbf{u}\|, \tag{2.2}$$

where

$$D_\Omega = \sup_{\mathbf{u}} \{\|\mathbf{u}\|_{L^4(\Omega)} / \|\nabla \mathbf{u}\|^2_{L^2(\Omega)} | \, \mathbf{u}, \quad \nabla \mathbf{u} \in L^2(\Omega)\},$$

the solution to the steady Navier–Stokes equations is unique. Therefore, for those cases, problem (2.1) is well posed.

The objective function in (2.1) is the rate of energy dissipated in the fluid; as we pointed out in the previous paper (Pironneau 1973), if Γ and $\mathbf{g}$ are as in figure 1, and if the surfaces S of $\mathscr{S}$ are 'centrally located' and small in Γ, the solution of (2.1) will be the element of $\mathscr{S}$ with smallest drag. The introduction of Γ and $\mathbf{g}$ is only an artifice to avoid an unbounded domain, and the convergence of the integral to be considered.

It is theoretically possible to study (2.1) when $\mathbf{u}$ is described by the full (time-dependent) Navier–Stokes equations. In fact, results very similar to those in this paper can probably be obtained. However, since this paper is the mathematical background for a numerical solution of the problem, and since numerical solution of unsteady flows is extremely difficult, we shall restrict our attention to the

steady cases. In practice this implies that our bodies S must be such that the boundary layer does not separate.

By choosing $\mathscr{S}$ properly, we investigate the following four problems.

Problem 1. $\mathscr{S}_1 = \{S|$ volume enclosed by $S = 1\}$: minimum-drag body of given volume.

Problem 2. $\mathscr{S}_2 = \{S|$ surface area of $S = 1\}$: minimum-drag body of given surface area.

Problem 3. $\mathscr{S}_3 = \{S|\ D \subset$ interior of $S\}$: minimum-drag body which contains a given object D.

Problem 4. $\mathscr{S}_4 = \{S \in \mathscr{S}_i| \int_S (2\boldsymbol{\sigma} - p\mathbf{I})\ \mathbf{J}.d\mathbf{s} = d\}$: minimum-drag body of the above kind ($i = 1, 2$, or 3) of given lift d in the direction $\mathbf{J}$.

3. Results

Theorem 1. If $S \in \mathscr{S}$ is parameterized by $\mathbf{s}$, i.e. $S = \{\boldsymbol{\xi}(\mathbf{s})|\mathbf{s} \in [0, 1]^{n-1}\}$, and S' is a surface in $\mathscr{S}$ 'close' to S and parameterized by $\mathbf{s}$, i.e.

$$S' = \{\boldsymbol{\xi}'(\mathbf{s})|\boldsymbol{\xi}'(\mathbf{s}) = \boldsymbol{\xi}(\mathbf{s}) + \mathbf{n}(\mathbf{s})\,\alpha(\mathbf{s}); \mathbf{s} \in [0, 1]^{n-1}\},$$

where $\mathbf{n}(\mathbf{s})$ is the normal to S at $\mathbf{s}$, then the change δE in energy dissipation

$$E(S) = \frac{2}{\nu}\int_{\Omega_S} \sigma_{ij}\sigma_{ij}\,d\Omega$$

due to the difference between S' and S is

$$\delta E = E(S') - E(S) = \nu\int_S \alpha(\mathbf{s})\left(\left\|\frac{\partial \mathbf{u}}{\partial n}\right\|^2 + 2\frac{\partial \mathbf{w}}{\partial n}.\frac{\partial \mathbf{u}}{\partial n}\right)dS + o(\alpha(\ )). \tag{3.1}$$

Here $\mathbf{w}$, together with q, is the solution of

$$\nu\nabla^2\mathbf{w} - \nabla\mathbf{u}.\mathbf{w} + \mathbf{u}.\nabla\mathbf{w} = -\mathbf{u}.\nabla\mathbf{u} + \nabla q;\ \nabla.\mathbf{w} = 0 \text{ in } \Omega_S;\dagger\ \mathbf{w}|_S = \mathbf{w}|_\Gamma = 0 \tag{3.2}$$

and $o(\alpha(\ ))$ is defined by

$$\lim_{\lambda \to 0} \lambda^{-1} o(\lambda\alpha(\ )) = 0.$$

The proof of theorem 1 is lengthy, hence we defer it until the end of the paper. We obtain the following corollaries.

Corollary 1. The solution S to problem 1 must be such that

$$\left\|\frac{\partial \mathbf{u}}{\partial n}\right\|^2 + 2\frac{\partial \mathbf{w}}{\partial n}.\frac{\partial \mathbf{u}}{\partial n} = \text{constant almost everywhere on } S. \tag{3.3}$$

Corollary 2. The solution S to problem 2 must be such that

$$R(\mathbf{s})\left(\left\|\frac{\partial \mathbf{u}}{\partial n}\right\|^2 + 2\frac{\partial \mathbf{u}}{\partial n}.\frac{\partial \mathbf{w}}{\partial n}\right) = \text{constant almost everywhere on } S, \tag{3.4}$$

where $R(\mathbf{s})$ is the radius of curvature of S at $\mathbf{s}$.

† $(\mathbf{w}, q)$ can be shown to exist by the same method as that used by Ladyzhenskaya (1963) for $\mathbf{u}$. Uniqueness is guaranteed by (2.2).

COROLLARY 3.† The solution S to problem 3 must be such that

$$\left\|\frac{\partial \mathbf{u}}{\partial n}\right\|^2 + 2\frac{\partial \mathbf{w}}{\partial n}\cdot\frac{\partial \mathbf{u}}{\partial n} = 0 \quad \text{on parts of } S \text{ which do not touch } D, \tag{3.5}$$

$$\left\|\frac{\partial \mathbf{u}}{\partial n}\right\|^2 + 2\frac{\partial \mathbf{w}}{\partial n}\cdot\frac{\partial \mathbf{u}}{\partial n} \geqslant 0 \quad \text{on parts of } S \text{ which do touch } D. \tag{3.6}$$

The proofs of these corollaries are straightforward (see Pironneau 1973); it suffices to choose S' as S with two small humps, one positive and one negative, such that $S' \in \mathscr{S}$.

Similar results for problem 4 are more difficult to obtain; we need the following lemma.

LEMMA 1. Let $d(S) = \int_S (2\boldsymbol{\sigma}^u - p\mathbf{I})\,\mathbf{J}\,.\,d\mathbf{S}$. If S' and S are as in theorem 1, then

$$d(S') - d(S) = -\nu \int \frac{\partial \mathbf{u}}{\partial n}\cdot\frac{\partial \mathbf{t}}{\partial n}\,\alpha(\mathbf{s})\,dS + o(\alpha(\ )), \tag{3.7}$$

where $\sigma^u_{i,j} = \frac{1}{2}\nu(\partial u_i/\partial x_j + \partial u_j/\partial x_i)$, and $\mathbf{t}$ is the solution of

$$\nu\nabla^2\mathbf{t} - \nabla\mathbf{u}\,.\,\mathbf{t} + \mathbf{u}\,.\,\nabla\mathbf{t} = \nabla\tau, \quad \nabla\,.\,\mathbf{t} = 0; \quad \mathbf{t}|_S = \mathbf{J}, \quad \mathbf{t}|_\Gamma = 0.$$

Proof. The correct proof is nearly as long as the proof of theorem 1. We shall give a heuristic argument below; it can be justified in the same way as in theorem 1. Thence, let $\mathbf{t}$ be an extension in Ω_S of $\mathbf{J}|_S$ with $\nabla\,.\,\mathbf{t} = 0$ and $\mathbf{t}|_\Gamma = 0$. Using integration by parts, it is readily shown that

$$-\int_S \boldsymbol{\sigma}\mathbf{J}\,.\,d\mathbf{S} = \int_{\Omega_S} \frac{1}{\nu}\sigma^u_{ij}\sigma^t_{ij}\,d\Omega + \int_{\Omega_S}\frac{\nu}{2}\nabla^2\mathbf{u}\,.\,\mathbf{t}\,d\Omega, \tag{3.8}$$

where $\sigma^t_{ij} = \frac{1}{2}\nu(\partial t_i/\partial x_j + \partial t_j/\partial x_i)$. Hence

$$d(S') - d(S) = -\int_{\Omega_S}\left\{\left[\frac{2}{\nu}\sigma^{u'}_{ij}\sigma^{t'}_{ij} + (\nu\nabla^2\mathbf{u}' - \nabla p')\,.\,\mathbf{t}'\right] - \left[\frac{2}{\nu}\sigma^u_{ij}\sigma^t_{ij} + (\nu\nabla^2\mathbf{u} - \nabla p)\,.\,\mathbf{t}\right]\right\}d\Omega$$
$$- \int_{\delta\Omega}\left[\frac{2}{\nu}\,\sigma^{u'}_{ij}\,\sigma^{t'}_{ij} + (\nu\nabla^2\mathbf{u}' - \nabla p')\,.\,\mathbf{t}'\right]d\Omega,$$

where $\delta\Omega = (\Omega_{S'} - \Omega_{S'} \cap \Omega_S) \cup (\Omega_S - \Omega_{S'} \cap \Omega_S)$.

Let $\delta\mathbf{u} = \mathbf{u}' - \mathbf{u}$ and $\delta\mathbf{t} = \mathbf{t}' - \mathbf{t}$. If we neglect the terms in $\delta\mathbf{u}\,.\,\delta\mathbf{t}$ in the first integral, and if we replace the second integral by a surface integral (because the domain of integration is a narrow strip of thickness $\alpha(\ )$), we find that

$$d(S') - d(S) = -\int_{\Omega_S}\left[\frac{2}{\nu}(\sigma^{\delta u}_{ij}\sigma^t_{ij} + \sigma^u_{ij}\sigma^{\delta t}_{ij}) + (\nu\nabla^2\delta\mathbf{u} - \nabla\delta p)\,.\,\mathbf{t} + (\nu\nabla^2\mathbf{u} - \nabla p)\,.\,\delta\mathbf{t}\right]d\Omega$$
$$+ \int_S\left[\frac{2}{\nu}\sigma^u_{ij}\sigma^t_{ij} + (\nu\nabla^2\mathbf{u} - \nabla p)\,.\,\mathbf{t}\right]\alpha(\mathbf{s})\,d\mathbf{s} + o(\alpha).$$

† Note that at zero Reynolds number $S = D$ is the solution of problem 3, the inequality (3.6) being automatically satisfied ($\mathbf{w} = 0$). However, this may no longer be the case at high Reynolds number.

Let I be the first integral ; using (3.8) again,

$$I = \int_S (2\boldsymbol{\sigma}^u - p\mathbf{I})\,\delta\mathbf{t}.d\mathbf{S} + \int_S 2\boldsymbol{\sigma}^t\,\delta\mathbf{u}.d\mathbf{S} + \int_{\Omega_S} \nu\nabla^2\mathbf{t}.\delta\mathbf{u}\,d\Omega - \int_{\Omega_S} (\nu\nabla^2\delta\mathbf{u} - \nabla\delta p).\mathbf{t}\,d\Omega.$$

Now, $\delta\mathbf{u}$ satisfies $\nu\nabla^2\delta\mathbf{u} - \mathbf{u}.\nabla\delta\mathbf{u} - \delta\mathbf{u}.\nabla\mathbf{u} - \delta\mathbf{u}.\nabla\delta\mathbf{u} = \nabla\delta p$. Again we can show that

$$\int_{\Omega_S} \delta\mathbf{u}.\nabla\delta\mathbf{u}.\mathbf{t}\,d\Omega$$

is of second order in $\alpha(\ )$. Thence

$$I = \int_S (2\boldsymbol{\sigma}^u\delta\mathbf{t} + 2\nabla^t\delta\mathbf{u} - p\,\delta\mathbf{t}).d\mathbf{S} + \int_{\Omega_S} (\nu\nabla^2\mathbf{t} - \mathbf{u}.\nabla\mathbf{t} + \nabla\mathbf{u}.\mathbf{t}).\delta\mathbf{u}\,d\Omega + \sigma(\alpha),$$

where we have used

$$\int_{\Omega_S} (\mathbf{u}.\nabla\delta\mathbf{u}).\mathbf{t}\,d\Omega = -\int_{\Omega_S} (\mathbf{u}.\nabla\mathbf{t}).\delta\mathbf{u}\,d\Omega.$$

Hence, if we choose $\mathbf{t}$ so that $\nu\nabla^2\mathbf{t} + \nabla\mathbf{u}.\mathbf{t} - \mathbf{u}.\nabla\mathbf{t} = \nabla\tau$, we find that

$$d(S') - d(S) = \int_S (2\boldsymbol{\sigma}^u\delta\mathbf{t} + 2\boldsymbol{\sigma}^t\delta\mathbf{u} - p\,\delta\mathbf{t} - \tau\delta\mathbf{u}).\,d\mathbf{S} + o(\alpha) - \int_S \left\{\frac{2}{\nu}\,\sigma^u_{ij}\sigma^t_{ij} + (\nu\nabla^2\mathbf{u} - \nabla p).\mathbf{t}\right\} dS.$$

Now, by definition, $\delta\mathbf{u}|_S = \mathbf{u}'|_S$, and by a Taylor expansion

$$\mathbf{u}'|_S = \mathbf{u}'|_{S'} - \alpha|\partial\mathbf{u}'/\partial n|_{S'} + o(\alpha),$$

but $\mathbf{u}'|_{S'} = 0$, by definition. Therefore

$$\delta\mathbf{u}|_S = -\alpha\left.\frac{\partial\mathbf{u}}{\partial n}\right|_S + o(\alpha), \quad \delta\mathbf{t}|_S = -\alpha\,\frac{\partial\mathbf{t}}{\partial n} + o(\alpha).$$

Now $\qquad \delta\mathbf{t}.d\mathbf{S} = -\alpha\dfrac{\partial\mathbf{t}}{\partial n}.d\mathbf{S} + o(\alpha) = o(\alpha), \quad \delta\mathbf{u}.d\mathbf{S} = o(\alpha).$

Hence, since $\mathbf{u}.\nabla\mathbf{u}|_S = 0$

$$d(S') - d(S) = -2\int_S \left(\boldsymbol{\sigma}^t\,\frac{\partial\mathbf{u}}{\partial n}.\mathbf{n} + \boldsymbol{\sigma}^u\,\frac{\partial\mathbf{t}}{\partial n}.\mathbf{n} - \frac{1}{\nu}\,\sigma^u_{ij}\sigma^t_{ij}\right)\alpha\,dS + o(\alpha),$$

which implies (3.7).

Corollary 4. The solution S to problem 4 must be such that there exists a λ such that

$$\left\|\frac{\partial\mathbf{u}}{\partial n}\right\|^2 + 2\,\frac{\partial\mathbf{u}}{\partial n}.\left(\frac{\partial\mathbf{w}}{\partial n} + \lambda\,\frac{\partial\mathbf{t}}{\partial n}\right) = \text{constant on } S,$$

where $\mathbf{w}$ and $\mathbf{t}$ are as in theorem 1 and lemma 1.

Proof. $-\frac{1}{2}\lambda$ is the Lagrangian multiplier associated with (3.7).

4. Discussion of the results

Knowledge of the derivative of a function f is valuable when seeking the minimum of the function. There are numerical methods, the so-called gradient methods, which generate sequences of the type $\{x_i\}$ $(i \geqslant 0)$, where

$$x_{i+1} = x_i + \lambda_i f'(x_i), \tag{4.1}$$

which converge to a minimum of $f(x)$. These methods can be extended to problems like (2.1); the 'derivative' of the objective function in (2.1) is simply the quantity in brackets under the integral in (3.1). Therefore, if the surface S' is obtained from $S = \{\boldsymbol{\xi}(\mathbf{s}) | \mathbf{s} \in [0, 1]^{n-1}\}$, by taking $\boldsymbol{\xi}'(\mathbf{s}) = \boldsymbol{\xi}(\mathbf{s}) + \alpha(\mathbf{s})\,\mathbf{m}(\mathbf{s})$, where

$$\alpha(\mathbf{s}) = -\lambda\left(\left\|\frac{\partial \mathbf{u}}{\partial n}\right\|^2 + 2\frac{\partial \mathbf{u}}{\partial n}\cdot\frac{\partial \mathbf{w}}{\partial n}\right) \tag{4.2}$$

and $\lambda \geqslant 0$ is small, S' has a smaller drag. For problem 4, α must be such that $S' \in \mathscr{S}_i$. Lemma 1 tells us that $-(\partial\mathbf{u}/\partial n).\partial\mathbf{t}/\partial n$ is the gradient of the constraint $d(S) = d$; for physical reasons this constraint can be replaced by $d(S) \geqslant d$, and a method of feasible directions can then be used (see, for example, Pironneau & Polak 1973).

Therefore it is reasonable to attempt a numerical study of problem (2.1). However, a numerical subroutine for solving the steady Navier–Stokes equations and determining $\mathbf{w}$ is needed. Note that it may be possible to measure

$$\left\|\frac{\partial \mathbf{u}}{\partial n}\right\|^2 + 2\frac{\partial \mathbf{u}}{\partial n}\cdot\frac{\partial \mathbf{w}}{\partial n}$$

in a turbomachine, by measuring the change in drag due to a small hump on S. Indeed, from theorem 1 if the volume v of the hump is small and if it does not induce separation, the change δF in drag is

$$\delta F = \nu\frac{v}{U_0}\left(\left\|\frac{\partial \mathbf{u}}{\partial n}\right\|^2 + 2\frac{\partial \mathbf{u}}{\partial n}\cdot\frac{\partial \mathbf{w}}{\partial n}\right). \tag{4.3}$$

We shall now study some cases where the determination of $\mathbf{w}$ is simpler.

PROPOSITION 1. At low Reynolds number $\partial\mathbf{w}/\partial n$ is small compared with $\partial\mathbf{u}/\partial n$.

Proof. Let W and U be typical values for $|\mathbf{w}|$ and $|\mathbf{u}|$ and let L be a typical length of the problem. Then from (3.2) we find that

$$\nu\frac{W}{L^2} + \frac{UW}{L} = \nu\frac{U}{L^2}.$$

Hence, $w/U = O(LU/\nu)$.

PROPOSITION 2. Let $\Omega_S \subset R^2$ and (s, n) be the tangential and normal co-ordinates close to S. At high Reynolds number and close to S, $\mathbf{w}$ is a solution of the boundary-layer equations

$$\nu\frac{\partial^3 w_s}{\partial n^3} + \frac{\partial^2 w_s}{\partial n^2}u_n + \frac{\partial^2 w_s}{\partial n\,\partial s}u_s + 2\frac{\partial u_n}{\partial n}\frac{\partial w_s}{\partial n} + 2\frac{\partial u_s}{\partial n}\frac{\partial w_s}{\partial s} = -\nu\frac{\partial^3 u_s}{\partial n^3}; \quad \frac{\partial w_s}{\partial s} + \frac{\partial w_n}{\partial n} = 0, \tag{4.4}$$

with the boundary conditions

$$w_s = w_n = 0 \quad \text{at} \quad n = 0, \quad \partial w_s/\partial s = w_s = 0 \quad \text{at} \quad n = \infty.$$

Proof. When $\nu = 0$ equations (3.2) become

$$\nabla \mathbf{u}.\mathbf{w} - \mathbf{u}.\nabla \mathbf{w} = \nabla q, \quad \nabla \mathbf{w} = 0; \quad \mathbf{w}|_{\partial\Omega_S} = 0. \tag{4.5}$$

However the solution to (4.5) does not represent $\mathbf{w}$ in the neighbourhood of S. Following Prandtl's approach, we shall look for an equation for $\mathbf{w}$ different from (4.5) which represents $\mathbf{w}$ in the neighbourhood of S. In (3.2), we make the following change of co-ordinates:

$$s' = s/L, \quad n' = nR^{\frac{1}{2}}/L, \quad u_{s'} = u_s/U, \quad u_{n'} = u_n R^{\frac{1}{2}}/U,$$
$$w_{s'} = w_s/U, \quad w_{n'} = w_n R^{\frac{1}{2}}/U, \quad q' = q/U^2,$$

where $R = LU/\nu$. Then (3.2) becomes

$$\frac{1}{R}\left(\frac{\partial^2(w_{s'}+u_{s'})}{\partial s'^2} + R\frac{\partial^2(w_{s'}+u_{s'})}{\partial n'^2}\right) - \frac{\partial u_{s'}}{\partial s'} w_{s'} - \frac{1}{R}\frac{\partial u_{n'}}{\partial s'} w_{n'} + u_{s'}\frac{\partial w_{s'}}{\partial s'} + u_{n'}\frac{\partial w_{s'}}{\partial n'} = \frac{\partial q'}{\partial s'},$$

$$R^{-\frac{3}{2}}\left(\frac{\partial^2(w_{n'}+u_{n'})}{\partial s'^2} + R\frac{\partial^2(w_{n'}+u_{n'})}{\partial n'^2}\right) - R^{\frac{1}{2}}\frac{\partial u_{s'}}{\partial n'} w_{s'}$$
$$- R^{-\frac{1}{2}}\left(\frac{\partial u_{n'}}{\partial n'} w_{n'} - u_{s'}\frac{\partial w_{n'}}{\partial s'} - \frac{\partial w_{n'}}{\partial n'} u_{n'}\right) = R^{\frac{1}{2}}\frac{\partial q'}{\partial n'},$$

$$\partial w_{s'}/\partial s' + \partial u_{n'}/\partial n = 0.$$

Therefore, when $R \to \infty$ we obtain

$$\frac{\partial^2(w_{s'}+u_{s'})}{\partial n'^2} + u_{s'}\frac{\partial w_{s'}}{\partial s'} + u_{n'}\frac{\partial w_{s'}}{\partial n'} - \frac{\partial u_{s'}}{\partial s'} w_{s'} = \frac{\partial q'}{\partial s'}, \tag{4.6}$$

$$-\frac{\partial u_{s'}}{\partial n'} w_{s'} = \frac{\partial q'}{\partial n'}, \tag{4.7}$$

$$\frac{\partial w_{s'}}{\partial s'} + \frac{\partial w_{n'}}{\partial n'} = 0, \tag{4.8}$$

which reduces to (4.4) after elimination of q'. The boundary conditions at ∞ derive from the fact that $\mathbf{w} = 0$ is an admissible solution in the inviscid region.

PROPOSITION 3. Around a wedge of angle $2\pi(1 - m/(m+1))$, or round a corner of angle $\pi(1-m/(m+1))$, $[\partial w_{s'}/\partial n]_S = \frac{1}{4}s'^{3\alpha-2}g''(0)$, where $\alpha = \frac{1}{2}(m+1)$, g is solution of

$$\left.\begin{aligned} g^{(4)} - 2\alpha f g^{(3)} - 2\alpha f' g'' + 4(2\alpha - 1) f'' g' &= -f^{(4)}; \\ g(0) = g'(0) = 0, \quad g'(\infty) = g''(\infty) &= 0 \end{aligned}\right\} \tag{4.9}$$

and where f is the solution to the Falkner–Skan equation.

Proof. Let ϕ be the stream function of $\mathbf{w}$ and ψ be the stream function of $\mathbf{u}$. From dimensional arguments, $\psi = s'^{\alpha} f(\frac{1}{2}n's'^{\alpha-1})$. The same dimensional analysis applies to ϕ: $\phi = s'^{\alpha} g(\frac{1}{2}h's'^{\alpha-1})$. Hence substituting $\mathbf{w}$ and $\mathbf{u}$ in (4.4) we obtain (4.9).

It is interesting to note that for $\alpha = \frac{1}{2}$ (flow past a flat plate) (4.9) can be integrated twice so that

$$g''(\eta) = -\tfrac{1}{2}f''(\eta) + c\frac{f'(\eta)}{f''(\eta)} + \frac{K}{f''(\eta)}.$$

Then, matching ϕ with the outer solution (4.5), we find that $g'(\infty) = 0$ implies that

$$c = -K = \left[2\int_0^\infty \frac{f'(\eta)-1}{f''(\eta)}\,d\eta\right]^{-1}.$$

COROLLARY. Around a wedge of angle 90°,

$$\left\|\frac{\partial \mathbf{u}}{\partial n}\right\|^2 + 2\frac{\partial \mathbf{w}}{\partial n}\cdot\frac{\partial \mathbf{u}}{\partial n}$$

is constant.

Proof. For an angle of 90°, $\alpha = \frac{2}{3}$ therefore $\partial\mathbf{w}/\partial n$ and $\partial\mathbf{u}/\partial n$ are constant.

5. Proof of theorem 1

Let $\sigma^S_{ij} = \frac{1}{2}\nu(\partial u^S_i/\partial x_j + \partial u^S_j/\partial x_i)$, where $\mathbf{u}^S = (u^S_1, \ldots, u^S_n)$ is the solution of the differential system on the right-hand side of (2.1). The change δE in dissipated energy is $\delta E = (2/\nu)\,\delta E'$, where

$$\delta E' = \int_{\Omega_{S'}} \sigma^{S'}_{ij}\sigma^{S}_{ij}\,d\Omega - \int_{\Omega_S} \sigma^S_{ij}\sigma^S_{ij}\,d\Omega. \tag{5.1}$$

Let $\Sigma = \{\boldsymbol{\xi}''(\mathbf{s})|\boldsymbol{\xi}''(\mathbf{s}) = \boldsymbol{\xi}(s) + \mathbf{n}(\mathbf{s})\,z(\mathbf{s});\, \mathbf{s}\in[0,1]^{n-1}\}$, where $z(\ )$ is a smooth function with

$$z(\mathbf{s}) \geqslant \max\{\alpha(\mathbf{s}), 0\}, \quad \mathbf{s}\in[0,1]^{n-1}.$$

Then (with self-explanatory notation)

$$\delta E' = \int_{\Omega_\Sigma} (2\sigma^S_{ij}\delta\sigma_{ij} + \delta\sigma_{ij}\delta\sigma_{ij})\,d\Omega + \int_{\Omega_{S'}-\Omega_\Sigma} \sigma^{S'}_{ij}\sigma^{S'}_{ij}\,d\Omega - \int_{\Omega_S-\Omega_\Sigma} \sigma^S_{ij}\sigma^S_{ij}\,d\Omega. \tag{5.2}$$

As a function of $\mathbf{x}\in\Omega$, $\sigma_{ij}(\ )$ is continuous almost everywhere (see, for example, theorem 5.5.7 in Ladyzhenskaya 1963). Therefore, from the mean-value theorem for integrals,

$$\delta E' = \int_{\Omega_\Sigma} (2\sigma^S_{ij}\delta\sigma_{ij} + \delta\sigma_{ij}\delta\sigma_{ij})\,d\Omega + \int_\Sigma [\sigma^{S'}_{ij}\sigma^{S'}_{ij}(z-\alpha) - \sigma^S_{ij}\sigma^S_{ij}z]\,d\Sigma + o(z-\alpha) + o(\alpha). \tag{5.3}$$

By integration by parts it is easy to show that

$$2\int_{\Omega_\Sigma} \sigma_{ij}\delta\sigma_{ij}\,d\Omega = -\nu^2\int_{\Omega_\Sigma} \nabla^2\mathbf{u}\,.\,\delta\mathbf{u}\,d\Omega - 2\nu\int_\Sigma \boldsymbol{\sigma}\,\delta\mathbf{u}\,.\,d\boldsymbol{\Sigma}. \tag{5.4}$$

By making use of $\nu\nabla^2\mathbf{u} = \mathbf{u}\,.\,\nabla\mathbf{u} + \nabla p$, and by integrating by parts the term $\nabla p\,.\,\delta\mathbf{u}$ (using the relation $\nabla\,.\,\mathbf{u} = 0$) (5.4) becomes

$$2\int_{\Omega_\Sigma} \sigma_{ij}\,\delta\sigma_{ij}\,d\Omega = -\nu\int_{\Omega_\Sigma} (\mathbf{u}\,.\,\nabla\mathbf{u})\,.\,\delta\mathbf{u}\,d\Omega - \nu\int_\Sigma (2\boldsymbol{\sigma} - p\mathbf{I})\,\delta\mathbf{u}\,.\,d\Sigma. \tag{5.5}$$

Now, let $(\mathbf{w}^\Sigma, q)$ be a solution of

$$\nu\nabla^2\mathbf{w}^\Sigma - (\nabla u_j)\,w^\Sigma_j + \frac{\partial\mathbf{w}}{\partial x_j}u_j = -\mathbf{u}\,.\,\nabla\mathbf{u} + \nabla q, \quad \nabla\,.\,\mathbf{w}^\Sigma = 0; \quad \mathbf{w}^\Sigma|_\Sigma = 0, \quad \mathbf{w}^\Sigma|_\Gamma = 0.$$

Then

$$2\int_{\Omega_\Sigma} \sigma_{ij}\,\delta\sigma_{ij}\,d\Omega = \nu\int_{\Omega_\Sigma} [\nu(\nabla^2\mathbf{w}^\Sigma)\,\delta\mathbf{u} - (\delta\mathbf{u}\,.\,\nabla\mathbf{u})\,.\,\mathbf{w}^\Sigma + (\mathbf{u}\,.\,\nabla\mathbf{w}^\Sigma)\,.\,\delta\mathbf{u} - \nabla q\,.\,\delta\mathbf{u}]\,d\Omega$$
$$-\nu\int_\Sigma (2\boldsymbol{\sigma} - p\mathbf{I})\,\delta\mathbf{u}\,.\,d\boldsymbol{\Sigma}. \quad (5.6)$$

From the definition of $\delta\mathbf{u}$, the first integral on the right-hand side above is equal to

$$+\int_{\Omega_\Sigma} (\delta\mathbf{u}\,.\,\nabla\delta\mathbf{u})\,.\,\mathbf{w}^\Sigma d\Omega + \int_{\Omega_\Sigma} \nabla(\delta p - q)\,.\,\delta\mathbf{u}\,d\Omega - \int_\Sigma \nu\frac{\partial\mathbf{w}^\Sigma}{\partial n}\,.\,\delta\mathbf{u}\,d\Sigma,$$

because $$\int_{\Omega_\Sigma} (\nabla^2\delta\mathbf{u})\,.\,\mathbf{w}^\Sigma d\Omega = \int_{\Omega_\Sigma} (\nabla^2\mathbf{w}^\Sigma)\,.\,\delta\mathbf{u}\,d\Omega + \int_\Sigma \frac{\partial\mathbf{w}^\Sigma}{\partial n}\,.\,\delta\mathbf{u}\,d\Sigma$$

and because $$-\int_{\Omega_\Sigma} (\mathbf{u}\,.\,\nabla\mathbf{w}^\Sigma)\,.\,\delta\mathbf{u}\,d\Omega = \int_{\Omega_\Sigma} (\mathbf{u}\,.\,\nabla\delta\mathbf{u})\,.\,\mathbf{w}^\Sigma d\Omega.$$

Hence, after having integrated by parts the term which contains $\delta p - q$, we find from (5.3) and (5.6), that

$$\delta E' = -\int_\Sigma 2\nu\boldsymbol{\sigma}^S\,\delta\mathbf{u}\,.\,d\boldsymbol{\Sigma} - \int_\Sigma \left(\sigma_{ij}^{S'}\sigma_{ij}^{S'}\alpha + \nu^2\frac{\partial\mathbf{w}^\Sigma}{\partial n}\,.\,\delta\mathbf{u}\right)d\Sigma + \delta^2 E, \quad (5.7)$$

where

$$\delta^2 E = \nu\int_{\Omega_\Sigma} (\delta\mathbf{u}\,.\,\nabla\delta\mathbf{u})\,.\,\mathbf{w}^\Sigma\,d\Omega + \int_{\Omega_\Sigma} \delta\sigma_{ij}\,\delta\sigma_{ij}\,d\Omega + \nu\int (p + q - \delta p)\,\delta\mathbf{u}\,.\,d\boldsymbol{\Sigma}$$
$$+\int_\Sigma (\sigma_{ij}^{S'}\sigma_{ij}^{S'} - \sigma_{ij}^{S}\sigma_{ij}^{S})\,z\,d\Sigma + o(z) + o(z-\alpha). \quad (5.8)$$

Later we shall prove that $\delta^2 E$ is small compared with the other terms in (5.7).

We now evaluate $\delta\mathbf{u}|_\Sigma$, as in the previous paper, by a Taylor expansion, which is legitimate because $\mathbf{u}^S \in C^2(\Omega_{S'})$ and $\mathbf{u}^S \in C^2(\Omega_S)$. We have

$$\mathbf{u}^{S'}|_\Sigma = \mathbf{u}^{S'}|_{S'} + \frac{\partial\mathbf{u}^{S'}}{\partial n}\bigg|_{S'}(z-\alpha) + o(z-\alpha),$$

$$\mathbf{u}^{S}|_\Sigma = \mathbf{u}^{S}|_{S} + \frac{\partial\mathbf{u}^{S}}{\partial n}\bigg|_{S} z + o(z).$$

A term of the form $o(z)$, say, in a vector equation represents a vector whose modulus is $o(z)$. Therefore, since $\mathbf{u}^{S'}|_{S'} = \mathbf{u}^S|_S = 0$

$$\delta\mathbf{u}|_\Sigma = -\frac{\partial\mathbf{u}^{S'}}{\partial n}\bigg|_{S'}\alpha + \left(\frac{\partial\mathbf{u}^{S'}}{\partial n}\bigg|_{S'} - \frac{\partial\mathbf{u}}{\partial n}\bigg|_{S}\right)z + o(z-\alpha) + o(z). \quad (5.9)$$

Hence, from (5.9), (5.7) becomes

$$\delta E' = \int_\Sigma \left(2\nu\boldsymbol{\sigma}\frac{\partial\mathbf{u}^{S'}}{\partial n}\bigg|_{S'}\,.\,\mathbf{n}_\Sigma - \sigma_{ij}^S\sigma_{ij}^S + \nu^2\frac{\partial\mathbf{w}^\Sigma}{\partial n}\,.\,\frac{\partial\mathbf{u}^{S'}}{\partial n}\bigg|_{S'}\right)\alpha\,d\Sigma + \delta^2 E + \delta^{2\prime}E, \quad (5.10)$$

where

$$\delta^{2\prime}E = \int_\Sigma \left(-2\nu\boldsymbol{\sigma}\,.\,\mathbf{n}_\Sigma - \nu^2\frac{\partial\mathbf{w}^\Sigma}{\partial n}\right)\cdot\left[\left(\frac{\partial\mathbf{u}^{S'}}{\partial n}\bigg|_{S'} - \frac{\partial\mathbf{u}^S}{\partial n}\bigg|_{S}\right)z + o(z-\alpha) + o(z)\right]d\Sigma. \quad (5.11)$$

Below, we shall prove that $\delta^2 E$ and $\delta^2 E'$ are small compared with the rest of δE, and that, when z, $\alpha \to 0$ we can erase the primes and replace S' by S in (5.10). Then, since

$$\sigma_{ij}\sigma_{ij}|_S = \frac{\nu^2}{2}\left\|\frac{\partial \mathbf{u}}{\partial n}\right\|^2, \quad \boldsymbol{\sigma}\frac{\partial \mathbf{u}}{\partial n}.\mathbf{n} = \frac{\nu}{2}\left\|\frac{\partial \mathbf{u}}{\partial n}\right\|^2,$$

the proof will be complete.

Roughly, $\delta^2 E + \delta^{2\prime} E$ is of second order in α for two reasons: first because $\mathbf{u}^S$ converges weakly to $\mathbf{u}^S$ when S' converges to S, and second because

$$\frac{\partial \mathbf{u}^S}{\partial n}.d\mathbf{S} = -\frac{\partial u_s^S}{\partial s}\,dS = 0$$

implies that

$$\int_S p\frac{\partial \mathbf{u}}{\partial n}.d\mathbf{S} = 0.$$

Now, we proceed carefully; we decompose $\delta^2 E + \delta^{2\prime} E$ into $A + B + C$, where

$$A = \int_{\Omega_\Sigma} (-\nu(\delta\mathbf{u}.\nabla\delta\mathbf{u}).\mathbf{w}^\Sigma + \delta\sigma_{ij}\delta\sigma_{ij})\,d\Omega, \tag{5.12}$$

$$B = -\nu\int_\Sigma (p+q-\delta p)\frac{\partial \mathbf{u}^{S'}}{\partial n_{S'}}\bigg|_{S'}.\alpha d\boldsymbol{\Sigma}, \tag{5.13}$$

$$C = \int_\Sigma \left\{(\sigma_{ij}^{S'}\sigma_{ij}^{S'} - \sigma_{ij}^{S}\sigma_{ij}^{S})\,z + \left\{[\nu(p+q-\delta p]\,\mathbf{I} - 2\nu\boldsymbol{\sigma}]\,\mathbf{n}_\Sigma - \nu^2\frac{\partial^2 \mathbf{w}^\Sigma}{\partial n}\right\} .\left(\frac{\partial \mathbf{u}^{S'}}{\partial n}\bigg|_{S'} - \frac{\partial \mathbf{u}^S}{\partial n}\bigg|_S\right) z\right\} d\Sigma. \tag{5.14}$$

The rest of $\delta^2 E + \delta^{2\prime} E$, i.e.,

$$o(\alpha) + o(z+\alpha) + \int_\Sigma \left\{\left[-\nu\boldsymbol{\sigma}\mathbf{n}_\Sigma + \nu^2\frac{\partial^2 \mathbf{w}^\Sigma}{\partial n}\right] + \nu(p+q-\delta p)\,(o(z) + o(z-\alpha))\right\} d\Sigma,$$

is clearly a second-order term in (z, α) once it has been shown that $\partial\mathbf{w}^\Sigma/\partial n$, and q and δp remain bounded when $S', \Sigma \to S$.

5.1. *A is a second-order term in the sense that* $\lim_{\lambda\to 0} \lambda^{-1} A(\lambda z, \lambda\alpha) = 0$

By elementary manipulation of inequalities, it is easy to show that there exists a $c_1 > 0$ with

$$|A(\alpha, z)| \leqslant c_1\|\delta\mathbf{u}\|^2_{H^1(\Omega_\Sigma)}. \tag{5.15†}$$

Hence

$$|A(\lambda\alpha, \lambda z)| \leqslant \lambda^2 c_1\|\delta\mathbf{u}/\lambda\|^2_{H^1(\Omega_\Sigma)}. \tag{5.16}$$

It was shown above that $\lambda^{-1}\delta\mathbf{u}(\lambda\alpha) = -[\partial\mathbf{u}^S/\partial n]_S\alpha + [o(\lambda(z-\alpha)) + o(\lambda z)]/\lambda$ (because $[\partial\mathbf{u}^{S'}/\partial n]_{S'}$ converges weakly to $[\partial\mathbf{u}^S/\partial n]_S$ as shown in the appendix). Let $\mathbf{a}$ be a solenoidal vector which takes the value $\lambda^{-1}/\delta\mathbf{u}\ (\lambda\alpha)$ on Σ' and is 0 on Γ, then from the above formula (and the fact, as shown later, that $[\partial\mathbf{u}^{S'}/\partial n]_{S'}$ remained bounded), $\|\mathbf{a}\|_{H^1(\Omega_\Sigma)}$ is uniformly bounded when $\lambda \to 0$.

Let $\delta\tilde{\mathbf{u}} = \lambda^{-1}\partial\mathbf{u}$. $\delta\tilde{\mathbf{u}}$ is the solution of

$$-\nu\nabla^2\delta\tilde{\mathbf{u}} + \mathbf{u}.\nabla\delta\tilde{\mathbf{u}} + \delta\tilde{\mathbf{u}}.\nabla\mathbf{u} + \lambda\,\delta\tilde{\mathbf{u}}.\nabla\delta\tilde{\mathbf{u}} = -\nabla\,\delta\tilde{p}, \quad \nabla.\delta\tilde{\mathbf{u}} = 0; \quad \delta\tilde{\mathbf{u}}|_{\partial\Omega_\Sigma} = \mathbf{a}|_{\partial\Omega_\Sigma}. \tag{5.17}$$

† $H^1(\Omega)$ is the Sobolev space of order 1 on Ω.

Let $\delta\mathbf{v} = \delta\tilde{\mathbf{v}} - \mathbf{a}$. If (5.7) is multiplied by $\delta\mathbf{v}$ and integrated by parts over Ω_Σ, it becomes

$$\int_{\Omega_\Sigma}\left\{\nu\frac{\partial}{\partial x_i}(\delta\mathbf{v}+\mathbf{a})\,.\,\frac{\partial\delta\mathbf{v}}{\partial x_i}+(\mathbf{u}\,.\,\nabla\mathbf{a})\,.\,\delta\mathbf{u}+[(\delta\mathbf{v}+\mathbf{a})\,.\,\nabla\mathbf{u}]\,.\,\delta\mathbf{v}\right.$$
$$\left.+\lambda[(\delta\mathbf{v}+\mathbf{a})\,.\,\nabla\mathbf{a}]\,.\,\delta\mathbf{v}\right\}d\Omega=0,$$

where we have used the formula

$$\int_\Omega(\mathbf{f}\,.\,\nabla\mathbf{g})\,.\,\mathbf{g}\,d\Omega=0\quad\text{for all}\quad\mathbf{f},\mathbf{g}\quad\text{such that}\quad\nabla\,.\,\mathbf{f}=0,\quad\mathbf{g}|_{\partial\Omega}=0.$$

Therefore, from the Schwartz and Poincaré inequalities†

$$\nu\|\delta\mathbf{v}\|^2_{H^1(\Omega_\Sigma)}\leqslant\|\delta\mathbf{v}\|_{H^1(\Omega_\Sigma)}[\nu\|a\|_{H^1(\Omega_\Sigma)}+\|\mathbf{u}\,.\,\nabla\mathbf{a}+\mathbf{a}\,.\,\nabla\mathbf{u}+\lambda\mathbf{a}\,.\,\nabla\mathbf{a}\|]$$
$$+\int_{\Omega_\Sigma}[\delta\mathbf{v}\,.\,\nabla(\mathbf{u}+\lambda\mathbf{a})]\,.\,\delta\mathbf{v}\,d\Omega.\quad(5.18)$$

As in Ladyzhenskaya (1963, formula (5.2.119)), we now show that (5.18) implies that there exists a λ_1 such that $\|\delta\mathbf{v}\|_{H^1(\Omega_\Sigma)}$ is uniformly bounded for all $\lambda\in(0,\lambda_1)$. Suppose that it is not true; then there exists a sequence $\{\lambda_i\}_{i\geqslant 0}$ such that $\lim_{i\to\infty}\lambda_i=\lambda_0$ and $\|\delta\mathbf{v}^i\|_{H^1(\Omega_\Sigma)}\to\infty$. Following the argument of Ladyzhenskaya, we find that (5.18) implies that

$$\nu\leqslant\lambda_0\sup_{\mathbf{v}}\left\{\int_\Omega[\mathbf{v}\,.\,\nabla(\mathbf{u}+\mathbf{b})]\,.\,\mathbf{v}\,d\Omega\,\middle|\,\|\mathbf{v}\|_{H^1}\leqslant 1\right\},$$

where $\mathbf{b}$ is a solenoidal extension of $\delta\mathbf{u}|_\Sigma$. Therefore if λ is small enough, the above formula is a contradiction, and hence $\|\delta\mathbf{v}\|_{H^1(\Omega_\Sigma)}$ must be bounded uniformly in λ. Then, (5.16) implies that

$$\lambda^{-1}|A(\lambda\alpha,\lambda z)\leqslant\lambda c_1\|\mathbf{a}+\delta\mathbf{v}\|_{H^1(\Omega_\Sigma)},$$

which implies the results.

5.2. *B is a second-order term in the sense that* $\lim_{\lambda\to0}\lambda^{-1}B(\lambda\alpha,\lambda z)=0$

Since $(\partial\mathbf{u}^\Sigma/\partial n_\Sigma)\,.\,d\mathbf{\Sigma}=0$ and since $[\partial\mathbf{u}^{S'}/\partial n_{S'}]_{S'}$ and $[\partial\mathbf{u}^\Sigma/\partial n_\Sigma]_\Sigma$ converge weakly to $[\partial\mathbf{u}^S/\partial n_S]_S$, the only part of B which is not obviously of second order is

$$\int_\Sigma\delta p\left(\frac{\partial\mathbf{u}^{S'}}{\partial n}\bigg|_{S'}-\frac{\partial\mathbf{u}^\Sigma}{\partial n}\bigg|_\Sigma\right).\,z\,d\mathbf{\Sigma}.$$

Let $\mathbf{a}'$ be an extension of

$$\left(\frac{\partial\mathbf{u}^{S'}}{\partial n}\bigg|_{S'}-\frac{\partial\mathbf{u}^\Sigma}{\partial n}\bigg|_\Sigma\right).\,\mathbf{n}_\Sigma z$$

† $\int_\Omega\|\nabla\mathbf{v}\|^2\,d\Omega\leqslant c\int_\Omega\|\mathbf{v}\|^2\,d\Omega\leqslant c'\int_\Omega\|\nabla\mathbf{v}\|^2\,d\Omega.$

in Ω_Σ. Then if (5.17) is multiplied by $\mathbf{a}'$ and integrated over Ω_Σ, it becomes

$$B(\lambda\alpha, \lambda z) = o(\lambda) + \lambda^2\nu\int_\Sigma \delta p\mathbf{a}' d\Sigma = o(\lambda) + \lambda^2\nu\int_{\Omega\Sigma} \mathbf{a}'.(\nu\nabla^2\delta\tilde{\mathbf{u}} + \mathbf{u}.\nabla\delta\tilde{\mathbf{u}} + \delta\tilde{\mathbf{u}}.\nabla\mathbf{u} + \lambda\delta\tilde{\mathbf{u}}.\nabla\delta\tilde{\mathbf{u}})\,d\Omega.$$

Since $\delta\tilde{\mathbf{u}}$ is uniformly bounded, it follows that there exists a B_0 with

$$|B(\lambda\alpha, \lambda z)| \leqslant \lambda^2 B_0 \quad \text{for all} \quad \lambda \in (0, \lambda_1).$$

5.3. *C is a second-order term in the sense that* $\lim_{\lambda\to 0} \lambda^{-1}C(\lambda\alpha, \lambda z) = 0$

This is an immediate consequence of the above calculation and of the fact that (i) $\sigma_{ij}^{S'}$ and $[\partial\mathbf{u}^{S'}/\partial n]_{S'}$ converge weakly to σ_{ij} and $[\partial\mathbf{u}^S/\partial n]_S$ and (ii) $\|\partial\mathbf{w}^\Sigma/\partial n\|_{L^2(\Sigma)}$ is uniformly bounded when $\Sigma \to S$.

6. Conclusions

We have obtained a certain amount of information about the disturbance in the fluid owing to a hump on the surface of a body. It would be interesting to connect these results with those obtained by Smith (1973). Our results are valid as long as the steady Navier–Stokes equations have a unique solution, that is, in practice, when the boundary layer does not separate. For other cases it is necessary to study the unsteady case. We have in fact assumed that the optimum profiles exist and are smooth. It now remains to solve the equations, a task that is not likely to be straightforward.

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Appendix. $[\partial\mathbf{u}^{S'}/\partial n]_{S'}$ converges weakly to $[\partial\mathbf{u}^S/\partial n]_S$ when S' converges to S

More precisely, we want to show that

$$\lim_{\substack{S'\to S\\ \Sigma\to S}} \int_\Sigma \phi\left(\frac{\partial\mathbf{u}^{S'}}{\partial n} - \frac{\partial\mathbf{u}^S}{\partial n}\right).d\boldsymbol{\Sigma} = 0 \quad \text{for all } \phi \text{ which are restrictions of functions of } C^1(\Omega_\Sigma). \tag{A 1}$$

The mapping A: $\mathbf{u}^S \to [\partial\mathbf{u}^S/\partial n_\Sigma]_\Sigma$ is continuous from $L^2(\Omega_\Sigma)$ into $H^{-\frac{3}{2}}(\Sigma)$. Therefore, since

$$\langle\phi, \partial(\mathbf{u}^{S'} - \mathbf{u}^S)/\partial n\rangle_{H^{\frac{3}{2}}(\Sigma)} = \langle\mathbf{u}^{S'} - \mathbf{u}^S, A^*\phi\rangle_{L^2(\Omega_\Sigma)}, \tag{A 2}$$

to prove (A 1) it suffices to show that $\mathbf{u}^{S'}$ converges weakly to $\mathbf{u}^S$ in $L^2(\Omega_\Sigma)$.

(*a*) $\mathbf{u}^{S'}$ *is uniformly bounded.* It is shown in Ladyzhenskaya (1963) that there exist c, $\mathbf{a}$ and Ω_δ such that

$$\|\mathbf{u}^{S'}\|_{H^1(\Omega_{S'})} \leqslant 2\|\mathbf{a}'\|_{H^1(\Omega_\delta)}(1 + c\|\mathbf{a}\|_{H^1(\Omega_\delta)}),$$

where $\mathbf{a}'$ is a solenoidal extension of $\mathbf{u}^{S'}|_{\partial\Omega_{S'}}$; Ω_δ is a strip of width δ around S' and Γ such that $\mathbf{a} \equiv 0$ outside Ω_δ. Since $\mathbf{u}^{S'}|_{S'} = 0$, Ω_δ is a strip along Γ only and

the right-hand side of (A 3) does not depend upon S'. Therefore if $\tilde{\mathbf{u}}^{S'}$ is the extension by zero of $\mathbf{u}^{S'}$ inside S',

$$\|\tilde{\mathbf{u}}^{S'}\|_{H^1(\Omega)} \leqslant 2\|\mathbf{a}'\|_{H^1(\Omega_\delta)}(1+c\|\mathbf{a}'\|)$$

for all S' close to S.

(*b*) *Let* $\mathbf{v}^S$ *be a weak limit of* $\tilde{\mathbf{u}}^{S'}-\mathbf{a}'$ (*it exists since* $\tilde{\mathbf{u}}^{S'}$ *is uniformly bounded*); *let* $\mathbf{v}^{S'} = \mathbf{u}^{S'}-\mathbf{a}'$. *If* $\mathbf{u}^{S'}$ *does not converge to* $\mathbf{u}^S$, *then there exists a* $\epsilon > 0$ *such that for* $\phi \in H^1(\Omega_S)$, *with* $\nabla . \boldsymbol{\phi} = 0$ *and* $\boldsymbol{\phi}|_{\partial\Omega_S} = 0$,

$$\int_{\Omega_S}\left(\frac{\partial v_i^S}{\partial x_j}\frac{\partial \phi_i}{\partial x_j} - \nabla^2\mathbf{a}' . \boldsymbol{\phi}\right)d\Omega + \int_{\Omega_S}((\mathbf{v}^S . \nabla\mathbf{a}' + \mathbf{a}' . \nabla\mathbf{v}^S . \boldsymbol{\phi} + (\mathbf{v}^S . \nabla\mathbf{v}^S) . \boldsymbol{\phi})\, d\Omega| \geqslant \epsilon.$$

Let $\tilde{\mathbf{v}}$ and $\boldsymbol{\phi}$ be the extensions of $\mathbf{v}$ and $\boldsymbol{\phi}$ by zero inside S. From the weak convergence of $\mathbf{v}^{S'}$ to $\mathbf{v}^S$

$$\left|\int_\Omega\left[\frac{\partial \tilde{v}_i^{S'}}{\partial x_j}\frac{\partial \tilde{\phi}_i}{\partial x_j} + (\tilde{\mathbf{v}}^{S'} . \nabla\mathbf{a}' + \mathbf{a}' . \nabla\tilde{\mathbf{v}}^{S'} + \tilde{\mathbf{v}}^{S'} . \nabla\tilde{\mathbf{v}}^{S'} - \nabla^2\mathbf{a}') . \boldsymbol{\phi}\right]d\Omega\right| \geqslant \frac{\epsilon}{2}.$$

Let $\boldsymbol{\phi}^{S'}$ be a sequence of elements of $H^1(\Omega)$ with $\nabla . \boldsymbol{\phi}^{S'} = 0$, $\boldsymbol{\phi}^{S'}|_{\partial\Omega_{S'}} = 0$ which converge *strongly* to $\boldsymbol{\phi}$; then

$$\left|\int_\Omega\left[\frac{\partial \tilde{v}_i^{S'}}{\partial x_j}\frac{\partial \tilde{\phi}_i^{S'}}{\partial x_j} + (\tilde{\mathbf{v}}^{S'} . \nabla\mathbf{a}' + \mathbf{a}' . \nabla\tilde{\mathbf{v}}^{S'} + \tilde{\mathbf{v}}^{S'} . \nabla\tilde{\mathbf{v}}^{S'} - \nabla^2\mathbf{a}') . \boldsymbol{\phi}^{S'}\right]d\Omega\right| \geqslant \frac{4}{\epsilon},$$

which contradicts the fact that $\tilde{\mathbf{v}}^{S'}+\mathbf{a}$ is a solution of the Navier–Stokes equation in $\Omega_{S'}$.

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